

# The normal closure of big Dehn twists, and plate spinning with rotating families

François Dahmani

March 28, 2017

## Abstract

We study the normal closure of a big power of one or several Dehn twists in a Mapping Class Group. We prove that it has a presentation whose relators consists only of commutators between twists of disjoint support, thus answering a question of Ivanov. Our method is to use the theory of projection complexes of Bestvina Bromberg and Fujiwara, together with the theory of rotating families, simultaneously on several spaces.

## Contents

<b>1</b>	<b>Composite projection systems</b>	<b>6</b>
1.1	Projection systems . . . . .	6
1.2	Composite projection systems . . . . .	8
1.3	Convexity . . . . .	11
<b>2</b>	<b>Composite rotating families and windmills</b>	<b>13</b>
2.1	Definition . . . . .	13
2.2	Osculations of two kinds . . . . .	15
2.3	The unfolding in the different coordinates . . . . .	16
2.4	Towers of windmills, and accessibility . . . . .	24
2.5	End of the proof of the main Theorem 2.2 . . . . .	26
<b>3</b>	<b>Conclusion, application to Dehn twists, and Theorem 1</b>	<b>26</b>

## Introduction

Consider a closed orientable surface  $\Sigma$  of negative Euler characteristic. The Mapping Class Group of  $\Sigma$ , denoted by  $\text{MCG}(\Sigma)$ , is the quotient of the group

of orientation-preserving homeomorphisms by the path-connected component of the identity. A classical theorem of Dehn and Nielsen indicates a natural isomorphism between this group and a subgroup of index 2 of the outer automorphism group of  $\pi_1(\Sigma)$ .

As Riemann uniformisation theorem makes  $\pi_1(\Sigma)$  act as a lattice on the hyperbolic plane, one can argue that  $\text{MCG}(\Sigma)$  is (in a sense) some hyperbolic analogue of  $SL_2(\mathbb{Z})$  which is of index 2 in the automorphism group of  $\mathbb{Z}^2$ , a lattice in the euclidean plane.

However, contrarily to  $SL_2(\mathbb{Z})$ , some elements of  $\text{MCG}(\Sigma)$  have large centraliser. For instance, consider a simple closed curve  $\alpha$  on  $\Sigma$ , a tubular neighborhood of it  $\alpha^{(t)} \simeq [-\epsilon, \epsilon] \times \alpha \hookrightarrow \Sigma$  and define a (simple) Dehn twist  $\tau$  as the identity in  $\Sigma \setminus \alpha^{(t)}$ , and as a full twist on  $\alpha^{(t)}$ , namely, identifying  $\alpha$  with  $S^1$ , the map  $[(\eta, e^{i\theta}) \mapsto (\eta, e^{i(\theta + \frac{(\eta+\epsilon)\pi}{\epsilon})})]$ . A Dehn twist will obviously commute with any mapping class whose support is disjoint from this tube, and therefore with a lot of other Dehn twists. By a theorem of Dehn,  $\text{MCG}(\Sigma)$  is generated by Dehn twists around simple closed curves, thus by an intricate set of generators linked by commutation relations, but also braid relations and lantern relations. These differences can lead to modify the expected analogy with the euclidean case in order to include  $SL_n(\mathbb{Z})$  for  $n \geq 3$  (generated by elementary matrices).

Thurston's classified mapping classes into three cases, those of finite order, those that are reducible in the sense that they have infinite order and that some nontrivial power preserves the homotopy class of a simple closed curve, and finally the pseudo-Anosov. The pseudo-Anosov mapping classes happen to be the hyperbolic isometries of an action of  $\text{MCG}(\Sigma)$  on an important graph, the curve graph of  $\Sigma$ , which is Gromov hyperbolic [MM]. They are, in many ways, the witnesses that some phenomena of rank one happen in  $\text{MCG}(\Sigma)$  that are similar to the structure of  $SL_2(\mathbb{Z})$ , and its action on the modular tree. On the other hand, Dehn twists are as reducible as it is possible to be. They are, or should be, the witnesses of some phenomena of higher rank, similar to the structure of  $SL_n(\mathbb{Z})$  for  $n \geq 3$ .

Here is an illustration of the difference of behaviors. If one considers a finite collection of pseudo-Anosov elements, one can (easily) show that, after taking powers, the group they generate is free. This is a simple ping-pong argument in the hyperbolic curve graph. If one considers a finite collection of Dehn twists around simple closed curves, then Koberda [K] proved the beautiful ping-pong result that the group generated by some powers of these Dehn twists is a right angled Artin group: a group whose presentation over

the given generating set is a collection of commutators, the obvious ones (two Dehn twists commute if their curves are disjoint).

The case of normal subgroups is our interest. If  $n \geq 3$ , by Margulis' normal subgroup theorem, all normal subgroups of  $SL_n(\mathbb{Z})$  are finite or of finite index. In  $SL_2(\mathbb{Z})$  it is not the case: this group is virtually free, and has uncountably many non-isomorphic quotients.

It is a natural question to ask whether (and how) these phenomena are seen in  $MCG(\Sigma)$ . What can be the normal closure of a power of a pseudo-Anosov, the normal closure of a power of a Dehn twist, and the group generated by all  $k$ -th powers of all simple Dehn twists? Farb and Ivanov asked this question in the case of a pseudo-Anosov (respectively [Fa, §2.4] and [I, §3]), attributing it to Long, McCarthy, and Penner. Ivanov also asked what he calls the deep relation question [I, §12], that is whether all relations among certain powers of Dehn twists must derive from obvious commutation relations.

In [DGO, §5], we answered the first question: there is an integer  $N = N(\Sigma)$  such that for all pseudo-Anosov mapping class  $\gamma$ , the normal closure  $\langle\langle \gamma^N \rangle\rangle_{MCG(\Sigma)}$  is free, and consists only of pseudo-Anosov elements and the identity. This is in line with what happens in  $SL_2(\mathbb{Z})$ , for all infinite order element.

We are interested in the question of the closure of a power of a Dehn twist, and in the group generated by certain powers of all (simple) Dehn twists, as in Ivanov's deep relation problem. A naive expectation along the lines of the analogy with  $SL_n(\mathbb{Z})$ , and the Margulis normal subgroup theorem, could be to expect such normal subgroups to be a finite index subgroup. Whereas it is the case for squares of Dehn twists [H], it is not the case for large powers (see [H], [Fu], [Cou, 6.17], see also [S] and [Mas] for the case of powers of half-Dehn twists on punctured spheres). Another expectation could be, in light of the finite-type situation, and ping pong arguments, to expect infinitely generated right angled Artin groups. Again, this is not the case in general (see [CLM] and [BM]; Brendle and Margalit proved restrictions on the automorphism group of certain of these normal subgroups, that forbid them to be right angled Artin groups). However, we indeed prove that there is no need of another relation than the obvious ones.

**Theorem 1.** *For every orientable closed surface  $\Sigma$ , there is  $N$  such that:*

- *for any Dehn twist  $\tau$ , the normal closure of  $\tau^N$  in the Mapping Class Group of  $\Sigma$  has a partially commutative presentation, built on an infinite set of generators that are conjugates of  $\tau^N$ , so that the relators*

*are commutations between pairs of conjugates of  $\tau^N$  that have disjoint underlying curves.*

- *the group generated by all  $N$ -th powers of all simple Dehn twists has a partially commutative presentation, built on an infinite set of generators that are  $N$ -th powers of Dehn twists, and whose relators are commutations between pairs of conjugates of the generators that have disjoint underlying curves.*

The difference with an infinitely generated right angled Artin group is that some elements in the commutator relators are not in the generating set, but merely conjugates of elements in the generating set. We recover that the normal closure is far from being of finite index in  $\text{MCG}(\Sigma)$ , for instance because it has abelianisation of infinite rank (the relators being in the derived subgroup of the free group over the set of generators).

In our point of view, this result above, and its departure from the complexity of normal subgroups of  $SL_n(\mathbb{Z})$  for  $n \geq 3$  (granted by Margulis normal subgroup theorem) reinforce [Fu, Cou] in witnessing a dent in the analogy between  $\text{MCG}(\Sigma)$  and  $SL_n(\mathbb{Z})$ . It also answers Ivanov's question on deep relations.

Let us discuss the proof of this theorem.

In [DGO] the structure of the normal closure of a big pseudo-Anosov was studied with the help of rotating families. Consider  $G$  a group acting by isometries on a space  $X$ . A rotating family in  $G$  on  $X$  is a collection of subgroups (the rotation groups), that is closed for conjugacy, such that each of them fix a certain point in  $X$  (thus inducing some kind of rotation around this point). Take  $\rho$  in one of these subgroups, fixing  $c$ . One may measure an analogue of the angle of rotation of  $\rho$  by taking  $x$  at distance 1 from  $c$ , and measuring the infimal length between  $x$  and  $\rho x$  of paths outside the ball of radius 1 around  $c$ . If  $X$  is Gromov-hyperbolic (for a small hyperbolicity constant), if the fixed point of the different rotation groups are sufficiently far from each other, and if the angles of rotations are sufficiently big, the group generated by all the rotation groups is a free product of a selection of them. In [DGO] we applied this theory to the action of  $\text{MCG}(\Sigma)$  on a cone-off of the curve graph of  $\Sigma$ . The rotation groups were the conjugates of the big pseudo-Anosov considered.

The rotating family argument can be explained as follows. One analyses the structure of groups generated by more and more rotation groups, to discover that they arrange as a sequence of free products. Starting from a quasi-convex set  $W$  (that will change over time) that is at first a small ball

around a single fixed point of a single rotation group, one make  $W$  grow until it (almost) touches another center of rotation, for some other group. Then one *unfolds*  $W$  into  $W'$  by taking its images by the group generated by the new rotations, and the rotation already with center in  $W$ . Because of hyperbolicity, and of largeness of angles of rotations involved, the resulting space is still quasi-convex, with the almost the same constant – with a little repair, it has the same quasi-convexity constant indeed. Actually  $W'$  has a structure of tree whose vertices are the images of  $W$  by the group, and the fixed points of the new rotations involved, thus giving by Bass-Serre duality the structure of free product of the group generated by rotations whose center are in  $W'$  (edge stabilizers are trivial since no element can fix two different centers of rotation). Then, one takes the new  $W$  as  $W'$  and do over. In the direct limit, the group generated by all rotations has been described as a free product of a selection of rotation groups.

In [BBF], Bestvina Bromberg and Fujiwara, using a system of subsurface projections, discovered that there is a normal finite index subgroup  $G_0$  of  $\text{MCG}(\Sigma)$  that acts on some spaces quasi-isometric to trees, and on which Dehn twists behave like large rotation subgroups. It has been observed by several people that this implies that the normal closure of a certain power of a Dehn twist in  $G_0$  is free, using the argument of [DGO]. However, it is far from obvious how to promote this structural feature to the normal closure in  $\text{MCG}(\Sigma)$ .

In this paper, we use several quasi-trees as above, one for each left coset of  $G_0$  in  $\text{MCG}(\Sigma)$ . The group  $G_0$  acts on each of them, but its action is twisted by the automorphism of  $G_0$  that is the conjugation by elements  $g_i, i = 1, \dots, m$  realising a transversal of  $G_0$  in  $\text{MCG}(\Sigma)$ . If  $\tau^N$  is a Dehn twist in  $G_0$ , the normal closure of  $\tau^N$  in  $\text{MCG}(\Sigma)$  equals the normal closure of the collection  $\{g_i \tau^N g_i^{-1}, i = 1, \dots, m\}$  in  $G_0$ . Each  $g_i \tau^N g_i^{-1}$  is a legitimate rotation on the quasi-tree associated to  $g_i$ .

The argument of [DGO] is then performed simultaneously on each of the  $m$  quasi-trees. Instead of one convex subset that grows, and gets unfolded in a hyperbolic space, we have  $m$  convex sets  $\mathcal{W}_1, \dots, \mathcal{W}_m$  in the  $m$  quasi-trees. Each of them is invariant by the group generated by the rotations around rotation points in all of them. One looks for a rotation point  $R$  that is nearby one of these sets, and in a certain sense, nearby all of them (although they do not live in the same quasi-trees, this still makes sense in the framework of projection systems). Then, one unfolds our convex sets in all coordinates  $i = 1, \dots, m$ . A funny phenomenon happens. The unfolding in the coordinate of  $R$  provides a nice tree, as the argument of [DGO], and the convexity of the result is quantitatively very good. This

tree gives the structure of the new group by Bass-Serre duality, and reveals that only commutation relations are involved. There is no reason that the unfolding in all other coordinates produce something resembling to a tree, and could in principle destroy the convexity of  $\mathcal{W}_j$ . However, using the properties of the projection system, we show that the result is still somehow convex (less convex than before though). The game is then to unfold in the different quasitrees at regular intervals of time in the process, and to control the degradation of the convexity so that the repair can wait until a new unfolding occurs. It is a game of plate spinning.

The quasi-trees that we will use come from projection complexes defined in [BBF]. We wrote the argument in this axiomatic language, to avoid dealing with useless hyperbolicity constants. In the end, even if the spaces are indeed quasi-trees, this fact does not appear in the argument. The axioms of projection systems are extensively used though, and they contain the information that the geometric space is a quasi-tree. We will thus prove a similar statement as Theorem 1, namely Theorem 2.2, that gives the structure of groups generated by composite rotating families. There is actually more information coming from this composite rotating family structure, as for instance the Greendlinger property (see Definition 2.3), that describes how an element in the group can be shortened in one coordinate of the composite projection system.

## 1 Composite projection systems

### 1.1 Projection systems

Let us recall a part of the axiomatic construction of [BBF].

**Definition 1.1.** ([BBF])

*A projection system is a set  $\mathbb{Y}$ , with a constant  $\theta > 0$ , and for each  $Y \in \mathbb{Y}$ , a function  $(d_Y^\pi : \mathbb{Y} \setminus \{Y\} \times \mathbb{Y} \setminus \{Y\} \rightarrow \mathbb{R}_+)$  satisfying four or five axioms:*

- *symmetry* ( $d_Y^\pi(X, Z) = d_Y^\pi(Z, X)$ ),
- *triangle inequality* ( $d_Y^\pi(X, Z) + d_Y^\pi(Z, W) \geq d_Y^\pi(X, W)$ ),
- *Behrstock inequality* ( $\min\{d_Y^\pi(X, Z), d_Z^\pi(X, Y)\} \leq \theta$ ),
- *properness* ( $\{Y, d_Y^\pi(X, Z) > \theta\}$  is finite).
- *In this work one also assume the separation* ( $d_Y^\pi(Z, Z) \leq \theta$ ).

Observe that if the axioms are true for some  $\theta$  they hold for all larger  $\theta$ .

From this rudimentary axiomatic set, Bestvina Bromberg and Fujiwara manage to extract meaningful geometry, by modifying the functions  $d_Y^\pi$  into some functions  $d_Y$ , that satisfy much more properties, usually encapsulated in the statement that the projection complex of  $\mathbb{Y}$ , for a suitable parameter  $K$  is a quasi-tree.

One should think of  $d_Y$  (or  $d_Y^\pi$ ) as an angular measure between  $X$  and  $Z$  seen from  $Y$ . The axioms fit in this viewpoint: the Behrstock inequality says that if the angle at  $Y$  between  $X$  and  $Z$  is large, then from the point of view of  $Z$ , the items  $Y$  and  $X$  look aligned.

Let us review very quickly the procedure of [BBF] to produce the functions  $d_Y$ . Given  $\theta$  for which the axioms hold, [BBF] define  $\mathcal{H}(X, Z)$  to be the set of pairs  $(X', Z')$  such that both  $d_X^\pi$  and  $d_Z^\pi$  between them is strictly larger than  $2\theta$ , and one also include the pairs  $(X, Z')$  if  $d_Z^\pi(X, Z') > 2\theta$ , symmetrically the pairs  $(X', Z)$  if  $d_X^\pi(X', Z) > 2\theta$ , and finally the pair  $(X, Z)$  itself.

Then  $d_Y(X, Z)$  is defined to be the infimum of  $d_Y^\pi$  over  $\mathcal{H}(X, Z)$ .

For all  $K$ ,  $\mathbb{Y}_K(X, Z)$  denotes the set  $\{Y, d_Y(X, Z) \geq K\}$ .

[BBF, Theorem 3.3] states that there exists  $\Theta$  and  $\kappa \geq \theta$  depending only on  $\theta$ , such that

- (Symmetry)  $d_Y(X, Z) = d_Y(Z, X)$  for all  $Y, X, Z$
- (Coarse equality)  $d_Y^\pi - \kappa \leq d_Y \leq d_Y^\pi$
- (Coarse triangle inequality)  $d_Y(X, Z) + d_Y(Z, W) \geq d_Y(X, W) - \kappa$
- (Behrstock inequality)  $\min\{d_Y(X, Z), d_X(Y, Z)\} \leq \kappa$
- (Properness)  $\{Y, d_Y(X, Z) > \Theta\}$  is finite
- (Monotonicity) If  $d_Y(X, Z) \geq \Theta$  then both  $d_W(X, Y), d_W(Z, Y)$  are less than  $d_W(X, Z)$ .
- (Order)  $\mathbb{Y}_\Theta(X, Z) \cup \{X, Z\}$  is totally ordered by an order  $\dot{<}$  such that  $X$  is lowest,  $Z$  is greatest, and if  $Y_0 \dot{<} Y_1 \dot{<} Y_2$ , then

$$d_{Y_1}(X, Z) - \kappa \leq d_{Y_1}(Y_0, Y_2) \leq d_{Y_1}(X, Z),$$

and

$$d_{Y_0}(Y_1, Y_2) \leq \kappa, \quad d_{Y_2}(Y_1, Y_0) \leq \kappa.$$

Then choosing  $K$  larger than  $\Theta$ , the projection complex  $\mathcal{P}_K(\mathbb{Y})$  is defined as follows: it is a graph whose vertices are the elements of  $\mathbb{Y}$  and where  $X, Z$  span an edge if and only if  $\mathbb{Y}_K(X, Z) = \emptyset$ . Then [BBF, Thm. 3.16] states that for sufficiently large  $K$ ,  $\mathcal{P}_K(\mathbb{Y})$  is connected and quasi-isometric to a tree for its path metric.

## 1.2 Composite projection systems

In this work, we are concern with a composite situation.

### 1.2.1 Definitions, and projection complexes

Let  $\mathbb{Y}_*$  be the disjoint union of finitely many countable sets  $\mathbb{Y}_1, \dots, \mathbb{Y}_m$ . Their indices  $i = 1, \dots, m$  are called the coordinates. Given  $Y \in \mathbb{Y}_*$ , denote by  $i(Y)$  its coordinate:  $Y \in \mathbb{Y}_{i(Y)}$ .

**Definition 1.2.** *A composite projection system on a countable set  $\mathbb{Y}_* = \sqcup_{i=1}^m \mathbb{Y}_i$  is the data of a constant  $\theta > 0$ , of a family of subsets  $\text{Act}(Y) \subset \mathbb{Y}_*$ ,  $Y \in \mathbb{Y}_*$  (the active set for  $Y$ ) such that  $\mathbb{Y}_{i(Y)} \subset \text{Act}(Y)$ , and of a family of functions  $d_Y^\pi : (\text{Act}(Y) \setminus \{Y\} \times \text{Act}(Y) \setminus \{Y\}) \rightarrow \mathbb{R}_+$ , satisfying the symmetry, the triangle inequality, the Behrstock inequality for  $\theta$  whenever both quantities are defined, the properness for  $\theta$  when restricted to each  $\mathbb{Y}_i$ , the separation for  $\theta$ , and also three other properties related to the map  $\text{Act}$ :*

- (symmetry in action)  $X \in \text{Act}(Y)$  if and only if  $Y \in \text{Act}(X)$ ,
- (closeness in inaction) if  $X \notin \text{Act}(Z)$ , for all  $Y \in \text{Act}(X) \cap \text{Act}(Z)$ ,  $d_Y^\pi(X, Z) \leq \theta$
- (finite filling) for all  $\mathcal{Z} \subset \mathbb{Y}_*$ , there is a finite collection of elements  $X_j$  in  $\mathcal{Z}$  such that  $\cup_j \text{Act}(X_j)$  covers  $\cup_{X \in \mathcal{Z}} \text{Act}(X)$ .

The closeness in inaction can be understood as a complement to Behrstock inequality: “if  $d_Y^\pi(X, Z) > \theta$ , then  $d_X^\pi(Y, Z)$  is defined and is less than  $\theta$ ”.

Applying [BBF] (as recalled in the previous subsection) we get, for each coordinate  $i \leq m$ , and for a suitable choice of  $\theta$ , a modified function  $d_Y : \mathbb{Y}_i \times \mathbb{Y}_i \rightarrow \mathbb{R}_+$ . This function is unfortunately not defined on  $\text{Act}(Y) \setminus \mathbb{Y}_i$ , but  $d_Y^\pi$  is defined on it, and thus we choose to define  $d_Y^\triangleleft(X, Z)$  to be  $d_Y$  if both  $X, Z$  are in  $\mathbb{Y}_i$  and  $d_Y^\pi$  otherwise.

We then define  $\mathbb{Y}_M^j(X, Z) = \{Y \in \mathbb{Y}_j \cap \text{Act}(X) \cap \text{Act}(Z), d_Y^\triangleleft(X, Z) \geq M\}$ . The elements  $X, Y, Z$  need not be in the same coordinate.



In the following we first choose  $\theta$  such that the construction of [BBF] applies for all coordinate  $\mathbb{Y}_i$ , and this provides the constants  $\Theta$  and  $\kappa$  (suitable for all coordinate).

Then we choose  $c_* > 1000(\Theta + \kappa)$ , and  $\Theta_P = c_* + 21m\kappa$ . One can choose  $K > \Theta_P$ . sufficiently large to get quasi-trees in all coordinate, but this is not important for us.

Finally, choose  $\Theta_{Rot} > 2c_* + 2\Theta_P + 20(\kappa + \Theta)$  for later purpose.

To keep track with the constants, it is worth keeping in mind that

$$\Theta_{Rot} \gg 2\Theta_P \gg 2c_* \gg 20(\Theta + \kappa) \gg \theta.$$

### 1.2.2 Group in the picture

An *automorphism* of composite projection system is a map  $\psi : \mathbb{Y}_* \rightarrow \mathbb{Y}_*$

- that induces a bijection on each  $\mathbb{Y}_i$ ,
- that sends  $\text{Act}(Y)$  to  $\text{Act}(\psi(Y))$ ,
- such that for all  $Y$ , and all  $X, Z \in \text{Act}(Y)$ ,  $d_Y^\triangleleft(X, Z) = d_{\psi(Y)}^\triangleleft(\psi(X), \psi(Z))$ .

A *rotation* around  $X \in \mathbb{Y}_*$  in a composite projection system  $\mathbb{Y}_*$  is an automorphism  $\psi$  such that  $\psi(X) = X$ , and such that for all  $Y \in \mathbb{Y}_* \setminus \text{Act}(X)$ , and for all  $W, Z \in \text{Act}(Y)$ ,  $\psi(Y) = Y$ , and  $d_Y^\triangleleft(W, Z) = d_Y^\triangleleft(W, \psi(Z))$ .

Let us now assume that a group  $G$  acts on the composite projection system by automorphisms.

Let us denote by  $G_X$  the stabilizer of  $X \in \mathbb{Y}_*$ .

We say that a subgroup  $\Gamma_X < G_X$  has *proper isotropy* if for all  $N > 0$  there is a finite subset  $F(N)$  of  $\Gamma_X$  such that if  $\gamma \in \Gamma_X \setminus F(N)$ , and if  $Y \in \text{Act}(X)$ , then  $d_X^\pi(Y, \gamma Y) > N$ .

### 1.2.3 Betweenness and orbit estimates

**Lemma 1.3.** (*Betweenness is transitive*)

If  $d_Y^\triangleleft(X, Z) > 2\kappa$  and  $d_Z^\triangleleft(Y, T) > 2\kappa$ , then  $Z$  is in  $\text{Act}(X)$  and  $d_Z^\triangleleft(X, T) \geq d_Z^\triangleleft(Y, T) - 2\kappa$ .

If  $d_Y^\triangleleft(X, Z) > 10\kappa$  and  $d_Z^\triangleleft(X, T) > 10\kappa$ , then  $d_Y^\triangleleft(X, T) \geq d_Y^\triangleleft(X, Z) - 2\kappa$ .

*Proof.* By Behrstock inequality, one has  $d_Z^\triangleleft(X, Y) \leq \kappa$  in both cases. For the first implication, by triangular inequality,  $d_Z^\triangleleft(X, T) \geq d_Z^\triangleleft(Y, T) - d_Z^\triangleleft(X, Y) - \kappa$ .

For the second implication,  $d_Z^\triangleleft(Y, T)$  is within  $2\kappa$  from  $d_Z^\triangleleft(X, T)$ . Behrstock inequality gives that  $d_Y^\triangleleft(Z, T) \leq \kappa$  and therefore  $d_Y^\triangleleft(X, T) \geq d_Y^\triangleleft(X, Z) - 2\kappa$ .  $\square$

**Lemma 1.4.** (*Orbit estimates, or transfert in a coordinate*)

Assume that  $\Gamma_X$  has proper isotropy.

For the finite subset  $F = F(10\kappa)$  of  $\Gamma_X$ , and for all  $Y \in \text{Act}(X)$ , and all  $X'$  that is either in  $\text{Act}(Y)$  or in  $\text{Act}(X)$ , and all  $\gamma \in \Gamma_X \setminus F$ , then either  $d_Y^\triangleleft(X', X) \leq \kappa$  or  $d_Y^\triangleleft(\gamma X', X) \leq \kappa$ .

*Proof.* Let us first treat the case of  $X' \in \text{Act}(Y)$ . If  $d_Y^\triangleleft(X', X) \leq \kappa$  we are done. Assume that  $d_Y^\triangleleft(X', X) > \kappa$ . By closeness in inaction,  $X' \in \text{Act}(X)$ , and by Behrstock inequality (and because  $\kappa \geq \theta$ ), one has  $d_X^\triangleleft(X', Y) \leq \kappa$ . By proper isotropy (and coarse triangle inequality),  $d_X^\triangleleft(\gamma X', Y) > 5\kappa$ . Thus, by Behrstock inequality again,  $d_Y^\triangleleft(\gamma X', X) \leq \kappa$ .

Now assume that  $X' \notin \text{Act}(Y)$ , but is in  $\text{Act}(X)$ . Since  $Y \in \text{Act}(X)$  we can measure  $d_X^\triangleleft(X', Y)$  and (since  $\Gamma_X$  preserves  $\text{Act}(X)$ ) also  $d_X^\triangleleft(\gamma X', Y)$ . By proper isotropy,  $d_X^\triangleleft(X', \gamma X') \geq 10\kappa$  and therefore at least one of the quantities  $d_X^\triangleleft(X', Y)$  and  $d_X^\triangleleft(\gamma X', Y)$  is larger than  $4\kappa$ . Assume for instance that  $d_X^\triangleleft(X', Y) \geq 4\kappa$ . Then by Behrstock inequality,  $d_Y^\triangleleft(X', X) \leq \kappa$ .  $\square$

Using this lemma four times, together with triangle inequality, one gets:

**Lemma 1.5.** (*Orbit estimates for proper isotropy*)

Let  $X_1, X_2, X'_1, X'_2$  such that  $X_1, X_2 \in \text{Act}(Y)$ . Assume that either  $X'_i$  is in  $\text{Act}(Y)$  or in  $\text{Act}(X_i)$ .

If the group  $\Gamma_{X_1}$  and  $\Gamma_{X_2}$  have proper isotropy, then for almost all elements  $\gamma_1 \in \Gamma_{X_1}$  and  $\gamma_2 \in \Gamma_{X_2}$ , one has

$$d_Y^\triangleleft(\gamma_1(X'_1), \gamma_2(X'_2)) - 4\kappa \leq d_Y^\triangleleft(X_1, X_2) \leq d_Y^\triangleleft(\gamma_1(X'_1), \gamma_2(X'_2)) + 4\kappa.$$

Recall that we chose  $K > 2\Theta + \kappa$ .

**Proposition 1.6.** (*Ellipticity*)

Given  $X \in \mathbb{Y}_*$ , and any  $j \leq m$ , the group  $G_X$  has an orbit in  $\mathcal{P}_K(\mathbb{Y}_j)$  of diameter at most 1.

*Proof.* If  $j = i(X)$ , and more generally, if  $G_X$  fixes an element  $Y \in \mathbb{Y}_j$ , it is obvious. Assume then that  $\mathbb{Y}_j \subset \text{Act}(X)$ .

The group  $G_X$  preserves the set  $\{Z \in \mathbb{Y}_j, \mathbb{Y}_{K_0}^j(X, Z) = \emptyset\}$  for any  $K_0$  hence for  $K_0 = (K - \kappa)/2 \geq \Theta$ . Consider  $Z_a, Z_b$  in this set, we claim that  $\mathbb{Y}_K^j(Z_a, Z_b)$  is empty. Assume  $Y \in \mathbb{Y}_K^j(Z_a, Z_b)$ . Since  $Y \in \text{Act}(X)$  we can consider  $d_Y^\triangleleft(Z_a, X)$  and  $d_Y^\triangleleft(Z_b, X)$ . By triangle inequality,  $d_Y^\triangleleft(Z_a, X) + d_Y^\triangleleft(Z_b, X) \geq d_Y^\triangleleft(Z_a, Z_b) - \kappa \geq K - \kappa$ . Thus, one of them needs to be larger than  $(K - \kappa)/2$  hence  $Y$  is either in  $\mathbb{Y}_{K_0}(X, Z_a)$  or in  $\mathbb{Y}_{K_0}(X, Z_b)$ , and this is a contradiction to our assumption.  $\square$

**Proposition 1.7.** (*Induced orders*)

Consider  $X, Z \in \mathbb{Y}_*$ , with  $Z \in \text{Act}(X)$ . Assume that  $\Gamma_X, \Gamma_Z$  are subgroups of  $G_X, G_Z$  with proper isotropy.

For all  $i \leq m$ , for all  $M \geq \Theta + 12\kappa$ , the set  $\mathbb{Y}_M^i(X, Z)$  is finite, and carries a partial order  $\dot{<}$ , that is given by the order of  $\mathbb{Y}_{M-4\kappa}^i(\gamma_X(X^i), \gamma_Z(Z^i))$ , for arbitrary  $X^i, Z^i$ , in  $\mathbb{Y}_i$ , and almost all  $\gamma_X \in \Gamma_X$  and  $\gamma_Z \in \Gamma_Z$ .

*Proof.* Let us first check that the set is finite. We may assume that there are  $X^i \in \text{Act}(X) \cap \mathbb{Y}_i$  and  $Z^i \in \text{Act}(Z) \cap \mathbb{Y}_i$ , otherwise  $\mathbb{Y}_M^i(X, Z)$  is empty. By Lemma 1.4, there exists  $\gamma_X \in \Gamma_X, \gamma_Z \in \Gamma_Z$  such that each  $Y \in \mathbb{Y}_M^i(X, Z)$  is in either one of the four sets  $\mathbb{Y}_{M-3\kappa}^i(\eta_X X^i, \eta_Z Z^i)$  for  $\eta_X \in \{1, \gamma_X\}$  and  $\eta_Z \in \{1, \gamma_Z\}$ . The union of these four sets is finite by properness axiom.

We now need to check that the order on  $\mathbb{Y}_{M-4\kappa}^i(\gamma_X(X^i), \gamma_Z(Z^i))$  includes all  $\mathbb{Y}_M^i(X, Z)$  and does not depend on the choice of the points  $X^i, Z^i$ . By the Lemma 1.5, for arbitrary choice of points, and for any  $Y \in \mathbb{Y}_M^i(X, Z)$ , there is a finite set of  $\Gamma_X$  and of  $\Gamma_Z$  such that for all elements  $\gamma_X, \gamma_Z$  outside these finite sets,  $Y \in \mathbb{Y}_{M-4\kappa}^i(\gamma_X X^i, \gamma_Z Z^i)$  (the finite sets depend on the choice of  $X^i, Z^i$  though). Since  $\mathbb{Y}_M^i(X, Z)$  is finite, we may find a finite set of  $\Gamma_X$  and  $\Gamma_Z$  suitable for all of them. Thus, for almost all  $\gamma_X, \gamma_Z$ , all  $\mathbb{Y}_{M-4\kappa}^i(\gamma_X(X^i), \gamma_Z(Z^i))$  is ordered, and the order, once chosen the points  $X^i, Z^i$ , does not depend on  $\gamma_X, \gamma_Z$ .

Assume that for two different choices of points  $X^i, Z^i$ , namely  $(X_a^i, Z_a^i)$  and  $(X_b^i, Z_b^i)$ , the orders are different, and take  $Y_1, Y_2$  such that  $Y_1 \dot{<}_a Y_2$  for the first order, and  $Y_2 \dot{<}_b Y_1$  for the other.

$Y_1 \dot{<}_a Y_2$  means that  $d_{Y_1}(Y_2, \gamma_Z(Z_a^i)) \leq \kappa$ . By the orbit estimate,  $d_{Y_1}^{\leq}(Y_2, Z) \leq 5\kappa$  for suitable  $\gamma_Z$ .

$Y_2 \dot{<}_b Y_1$  means that  $d_{Y_1}(Y_2, \gamma_X(X_b^i)) \leq \kappa$ , and by the orbit estimate,  $d_{Y_1}^{\leq}(Y_2, X) \leq 5\kappa$ . Finally, by coarse triangular inequality,  $d_{Y_1}^{\leq}(Z, X) \leq 11\kappa$ , contradicting the assumption that  $Y_1$  is in  $\mathbb{Y}_M^i(X, Z)$ .  $\square$

### 1.3 Convexity

**Definition 1.8.** (*Convexity*)

Let  $L > 10\kappa$ . We say that a subset  $\mathcal{W} \subset \mathbb{Y}_*$  is  $L$ -convex if: for all  $i$ , for all  $X, Z \in \mathcal{W} \cap \mathbb{Y}_i$ , for all  $j$ , the set  $\mathbb{Y}_L^j(X, Z)$  is a subset of  $\mathcal{W}$ .

Let now  $\mathcal{L} = (L(1), \dots, L(m))$  be a  $m$ -tuple of positive numbers. A subset  $\mathcal{W}$  of  $\mathbb{Y}_*$  is said  $L$ -convex if for all  $X, Z \in \mathcal{W}$ , of same coordinate  $i(X) = i(Z)$ , and for all  $j$ , the set  $\mathbb{Y}_{L(j)}^j(X, Z)$  is a subset of  $\mathcal{W}$ .

Note that being  $L$ -convex, for  $L > 0$  is equivalent to being  $(L, \dots, L)$ -convex.

**Definition 1.9.** Let  $\mathcal{W} \subset \mathbb{Y}_*$  non-empty, and  $R \in \mathbb{Y}_* \setminus \mathcal{W}$  for which  $\text{Act}(R) \cap \mathcal{W}$  is non-empty. Let  $L \geq 10\kappa$ . Define  $\mathbb{Y}_L(\mathcal{W}, R)$  as the set of  $Y \in \mathbb{Y}_*$  satisfying the following.

- $Y \in \text{Act}(R)$
- $Y \notin \mathcal{W}$
- $\mathcal{W} \cap \text{Act}(R) \cap \text{Act}(Y)$  is non-empty, and for all  $X \in \mathcal{W} \cap \text{Act}(R) \cap \text{Act}(Y)$ , one has  $Y \in \mathbb{Y}_L^{i(Y)}(X, R)$ .

**Proposition 1.10.** If  $L \geq \Theta + 12\kappa$ , then for all  $R$  for which it is defined, the set  $\mathbb{Y}_L(\mathcal{W}, R)$  is finite.

*Proof.* From the definition,  $\mathbb{Y}_L(\mathcal{W}, R) \subset \bigcup_i \bigcap_{X \in \text{Act}(R) \cap \mathcal{W}} (\mathbb{Y}_L^i(X, R) \cup (\mathbb{Y}_i \setminus \text{Act}(X)))$ . By finite filling assumption on the projection system, there is a finite collection of elements  $X_j \in \mathcal{W} \cap \text{Act}(R)$  such that  $\bigcup_j \text{Act}(X_j)$  covers  $\bigcup_{\mathcal{W} \cap \text{Act}(R)} \text{Act}(X)$ .

In particular,  $\mathbb{Y}_L(\mathcal{W}, R)$  is inside a finite union of sets of the form  $\mathbb{Y}_L^i(X_j, R)$  which are finite by Proposition 1.7.  $\square$

**Proposition 1.11.** Assume that for all  $X \in \mathcal{W}$ ,  $\mathcal{W}$  is invariant by an infinite group  $\Gamma_X$  of rotations around  $X$ , with proper isotropy. Let  $L \geq \Theta + 12\kappa$ .

If  $\mathcal{W}$  is  $(L - 6\kappa)$ -convex, and if  $S \in \mathbb{Y}_L(\mathcal{W}, R)$  then  $\mathbb{Y}_L(\mathcal{W}, S) \subset \mathbb{Y}_{L-2\kappa}(\mathcal{W}, R)$ . Moreover, if  $\mathcal{W}'$  contains  $\mathcal{W}$ , then  $\mathbb{Y}_L(\mathcal{W}', R) \subset \mathbb{Y}_L(\mathcal{W}, R)$ .

*Proof.* Let  $Y \in \mathbb{Y}_L(\mathcal{W}, S)$  in coordinate  $i$ . There exists  $X \in \mathcal{W} \cap \text{Act}(Y) \cap \text{Act}(S)$  such that  $d_Y^\triangleleft(X, S) \geq L$ .

Assume that  $\tilde{X} \in \text{Act}(R) \cap \text{Act}(Y) \cap \mathcal{W}$ . If it is not in  $\text{Act}(S)$ , then  $d_Y^\triangleleft(\tilde{X}, S) < \kappa$  and  $d_Y^\triangleleft(\tilde{X}, X) > L - 2\kappa$ . Transferring  $\tilde{X}$  in the coordinate of  $X$  (by invariance under  $\Gamma_{\tilde{X}}$ ), one has  $d_Y^\triangleleft(\tilde{X}_{i(X)}, X) > L - 6\kappa$ . By convexity,  $Y \in \mathcal{W}$  though we assumed otherwise. Therefore,  $\tilde{X} \in \text{Act}(S)$ . Therefore, by definition of  $\mathbb{Y}_L(\mathcal{W}, S)$ , one has  $d_Y^\triangleleft(\tilde{X}, S) \geq L$ , but also  $d_S^\triangleleft(\tilde{X}, R) \geq L$ . It follows by transitivity of betweenness (Lemma 1.3) that  $d_Y^\triangleleft(\tilde{X}, R) \geq L - 2\kappa$ .

The second assertion is a direct consequence of the definition.  $\square$

**Proposition 1.12.** If  $\text{Act}(R) \cap \mathcal{W}$  is not empty, for all  $L \geq (2m + 12)\kappa + \Theta$ , there exists  $Z \in \mathbb{Y}_L(\mathcal{W}, R)$  such that  $\mathbb{Y}_{L-2m\kappa}(\mathcal{W}, Z) = \emptyset$ .

*Proof.* Let us say that  $R$  has  $k$   $L$ -links to  $\mathcal{W}$  if  $\{i, \mathbb{Y}_L(R, \mathcal{W}) \cap \mathbb{Y}_i \neq \emptyset\}$  has  $k$  elements.

For any such index  $i$ , take a minimal item  $Z_i$  in  $\mathbb{Y}_L(R, \mathcal{W}) \cap \mathbb{Y}_i$  for the order of Proposition 1.7. Then, by Proposition 1.11,  $\mathbb{Y}_{L-2\kappa}(\mathcal{W}, Z_i)$  is included in  $\mathbb{Y}_L(R, \mathcal{W})$ , thus  $Z_i$  has at most  $(k-1)(L-2\kappa)$ -links to  $\mathcal{W}$ .

Iterating this choice at most  $m$  times, we find an element  $Z$  that has no  $(L-2m\kappa)$ -links to  $\mathcal{W}$ . Therefore  $\mathbb{Y}_{L-2m\kappa}(\mathcal{W}, Z) = \emptyset$ .  $\square$

**Proposition 1.13.** *Let  $L \geq \Theta + 12\kappa$ . Consider  $\mathcal{W}$ , and assume it is  $L$ -convex, and that for all  $X \in \mathcal{W}$ , there is  $\Gamma_X < G_X$ , infinite, that leaves  $\mathcal{W}$  invariant and that has proper isotropy.*

*If  $\mathbb{Y}_{L'}(\mathcal{W}, R)$  is well defined and empty, then  $\mathcal{W} \cup \{R\}$  is  $(L + L' + 5\kappa)$ -convex.*

*Proof.* If  $\mathcal{W} \cap \mathbb{Y}_{i(R)}$  is empty, there is nothing to prove. We assume it is non-empty. Consider  $Y \in \mathbb{Y}_{L+L'+5\kappa}(R, X)$  for some  $X \in \mathcal{W} \cap \mathbb{Y}_{i(R)}$ , and assume that  $Y \notin \mathcal{W}$ . Notice that  $Y \in \text{Act}(R)$  though, and  $X \in \text{Act}(R)$  since they have same coordinate. Hence,  $X \in \mathcal{W} \cap \text{Act}(R) \cap \text{Act}(Y)$ .

Let  $X'$  be any other element of  $\mathcal{W} \cap \text{Act}(R) \cap \text{Act}(Y)$ . Transfert  $X'$  in the coordinate  $i = i(R)$ , inside  $\mathcal{W}$ , by  $\Gamma_{X'}$ . There exists  $X'_i \in \mathbb{Y}_i \cap \mathcal{W}$  such that  $d_Y^{\leq}(X', X'_i) \leq \kappa$ . But,  $\mathcal{W}$  being  $L$ -convex, one has  $d_Y^{\leq}(X', X'_i) \leq L$ . It follows by triangular inequality, that  $d_Y^{\leq}(R, X') \geq L' + 2\kappa$ . Since this is true for all  $X'$  as above, it follows that  $Y \in \mathbb{Y}_{L'+2\kappa}(\mathcal{W}, R)$ , contradicting our assumption.  $\square$

## 2 Composite rotating families and windmills

We proceed to adapt the rotating families study of [DGO] to the context of composite projection systems.

### 2.1 Definition

**Definition 2.1.** *(Composite rotating family)*

*A composite rotating family on a composite projection system, endowed with an action of a group  $G$  by isomorphisms, is a family of subgroups  $\Gamma_Y, Y \in \mathbb{Y}_*$  such that*

- *for all  $X \in \mathbb{Y}_*$ ,  $\Gamma_X < G_X = \text{Stab}_G(X)$ , is an infinite group of rotation around  $X$*

- for all  $g \in G$ , and all  $X \in \mathbb{Y}_*$ , one has  $\Gamma_{gX} = g\Gamma_X g^{-1}$
- if  $X \notin \text{Act}(Z)$  then  $\Gamma_X$  and  $\Gamma_Z$  commute,
- for all  $i$ , for all  $X, Y, Z \in \mathbb{Y}_i$ , if  $d_Y(X, Z) \leq \Theta_P$  then for all  $g \in \Gamma_Y \setminus \{1\}$ ,  $d_Y(X, gZ) \geq \Theta_{\text{Rot}}$ .

We will show the following.

**Theorem 2.2.** *Consider  $\mathbb{Y}_*$  a composite projection system. If  $\{\Gamma_Y, Y \in \mathbb{Y}_*\}$  is a composite rotating family for sufficiently large  $\Theta_{\text{Rot}}$ , then the group  $\Gamma_{\text{Rot}}$  generated by the union of all  $\Gamma_Y$  (for  $Y \in \mathbb{Y}_*$ ) has a partially commutative presentation.*

*More precisely, there exists  $\mathcal{S} \subset \mathbb{Y}_*$ , and  $\mathcal{R}$  a set of words of the form  $[s, ws'w^{-1}]$  with  $w \in \langle \bigcup_{\mathcal{S}} \Gamma_S \rangle$  such that  $\Gamma_{\text{Rot}} \simeq \langle \bigcup_{\mathcal{S}} \Gamma_S \mid \mathcal{R} \rangle$  and such that  $[s, ws'w^{-1}] \in \mathcal{R}$  if and only if  $s \notin \text{Act}(ws')$ .*

*Moreover, for all  $\gamma \in \Gamma_{\text{Rot}} \setminus \{1\}$ , there is  $i(\gamma) \leq m$ , and  $R \in \mathbb{Y}_{i(\gamma)}$  such that for all  $X \in \mathbb{Y}_{i(\gamma)}$ ,  $d_R(X, \gamma X) > \Theta_{\text{Rot}} - 2\Theta_P - \kappa$ , and there is  $\gamma_s \in \Gamma_R$  such that  $d_R(X, \gamma_s \gamma X) \leq 2\Theta_P + 3\kappa$  (the pair  $(R, \gamma_s)$  is called a shortening pair for  $\gamma$ ).*

The last property is, in our point of view, an incarnation of the Greendlinger lemma, from the small cancellation theories. If one consider a relation  $\gamma$  of the quotient group, one can find in it a large part of a defining relation  $\gamma_s$ . Compare to [DGO, §5.1.3]

A major tool for analysing rotating families was the concept of windmills. We are going to use composite windmills.

Let us fix  $\mathcal{L}$  the  $m$ -tuple

$$\mathcal{L} = (c_* + 20(m-1)\kappa, c_* + 20(m-2)\kappa, \dots, c_* + 20\kappa, c_*).$$

Let  $\sigma$  be the cyclic shift on  $\mathbb{Z}/m\mathbb{Z}$ :  $\sigma(i) = (i-1)$ , and define  $\mathcal{L}_j = \sigma^{j-1}(\mathcal{L})$  obtained by shifting the coordinates of the  $m$ -tuple.

Thus  $\mathcal{L}_i$  reaches its maximum  $c_* + 20(m-1)\kappa$  on the coordinate  $i$ , minimal value  $c_*$  at  $i-1$ . Note that the maximum of  $\mathcal{L}$  is less than  $\Theta_P - \kappa$ .

**Definition 2.3.** *(Composite windmills)*

*A composite windmill is a collection  $(\mathcal{W}_1, \dots, \mathcal{W}_m, G_W, j_0)$  in which*

- $G_W$  is the subgroup of  $G$  generated by a set of subgroups  $\{\Gamma_Y, Y \in \bigcup_{i \in I_*} \mathcal{W}_i\}$  for  $I_*$  either  $\{1, \dots, m\}$  or  $\{1, \dots, m\} \setminus \{j_0\}$ ,
- $\mathcal{W}_i$  is a subset of  $\mathbb{Y}_i$  for all  $i$ , invariant under  $G_W$ ,

- $1 \leq j_0 \leq m$ ,
- $\bigcup_i \mathcal{W}_i$  is  $\mathcal{L}_{j_0}$ -convex.
- The group  $G_W$  has a partially commutative presentation, that is a presentation of the form

$$G \simeq \langle \mathcal{S} \mid \mathcal{R} \rangle$$

where  $\mathcal{S}$  is the union over a subset  $\mathcal{W}_*$  of  $\mathcal{W}$  of generating sets for  $\Gamma_X, X \in \mathcal{W}_*$ , and  $\mathcal{R}$  consists of words over the alphabet  $\mathcal{S} \cup \mathcal{S}^{-1}$  of the form  $[s, ws'w^{-1}]$  for  $w$  a word over  $\mathcal{S} \cup \mathcal{S}^{-1}$ . Moreover, if  $X, X' \in \mathcal{W}_*$  and  $s \in \Gamma_X, s' \in \Gamma_{X'}$ , the word  $[s, ws'w^{-1}]$  is in  $\mathcal{R}$  if and only if  $wX' \notin \text{Act}(X)$ .

- (Greendlinger property) for each  $\gamma \in G_W \setminus \{1\}$  there is  $i(\gamma) \leq m$ , and  $R \in \mathcal{W}_{i(\gamma)}$  such that for all  $X \in \mathbb{Y}_{i(\gamma)}$ ,  $d_R(X, \gamma X) > \Theta_{\text{Rot}} - 2\Theta_P - \kappa$ . Moreover, there is  $\gamma_s \in \Gamma_R$  such that  $d_R(X, \gamma_s \gamma X) \leq 2\Theta_P + 3\kappa$  (the pair  $(R, \gamma_s)$  is called a shortening pair for  $\gamma$ ).

We say that the composite windmill has full group if  $G_W$  is the subgroup of  $G$  generated by  $\{\Gamma_Y, Y \in \bigcup_{i=1}^m \mathcal{W}_i\}$ .

If we do not mention it, our windmills will be full. Only in specific circumstances do we need non-full windmills. Indeed, we will use the case of a non-full group only at most one time by coordinate, when initiating the process in each coordinate.

Since  $K > \max(\mathcal{L})$ , for each  $i$ ,  $\mathcal{W}_i$  is connected in  $\mathcal{P}_K(\mathbb{Y}_i)$ .

We say that a windmill  $\mathcal{W}'$  (with its representative set  $\mathcal{W}'_*$  used for the presentation of the definition) is *constructed over*  $\mathcal{W}$  if  $\mathcal{W} \subset \mathcal{W}'$  and if the set of representatives  $\mathcal{W}'_*$  contains the set of representatives  $\mathcal{W}_*$ .

## 2.2 Osculations of two kinds

- An osculator of type *gap* of a composite windmill  $(\mathcal{W}_1, \dots, \mathcal{W}_m, G_W, j_0)$  is an element  $R$  of  $\mathbb{Y}_{j_0} \setminus \mathcal{W}_{j_0}$  such that there exists  $i \leq m$ ,  $X_i, Z_i \in \mathcal{W}_i$ , that are in  $\text{Act}(R)$  and such that  $d_R^{\leq}(X, Z) > \frac{c_*}{2} - 20\kappa$ .
- An osculator of type *neighbor* of a composite windmill  $(\mathcal{W}_1, \dots, \mathcal{W}_m, G_W, j_0)$  is an element  $R$  of  $\mathbb{Y}_{j_0} \setminus \mathcal{W}_{j_0}$  such that  $\mathbb{Y}_{\frac{c_*}{2}}(\mathcal{W}, R) = \emptyset$ .

**Lemma 2.4.** Consider a composite windmill  $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_m, G_W, j_0)$ , assume that  $\mathcal{W}_{j_0} \neq \emptyset$ , and let  $R \in \mathbb{Y}_{j_0}$  be an osculator of type *gap*.

Let  $Y \in \mathbb{Y}_i$  in  $\text{Act}(R)$ . Then there exists  $X \in \mathcal{W}_{j_0}$  such that  $d_Y^{\leq}(X, R) \leq \kappa$ .

*Proof.* If  $R$  is an osculator of type gap, there are  $X', Z' \in \mathcal{W}_i$ , for some  $i$ , such that  $d_{R_1}^{\leq}(X', Z') > c_*/2 - 20\kappa$ .

Let  $X_0 \in \mathcal{W}_{j_0}$ , and consider its orbit under the groups  $\Gamma_{X'}$ , and  $\Gamma_{Z'}$ , which preserves  $\mathcal{W}_{j_0}$ . We may use Lemma 1.5 to find  $X'^{(j_0)}, Z'^{(j_0)}$  in these orbits, hence in  $\mathcal{W}_{j_0}$ , such that  $d_{R_1}^{\leq}(X'^{(j_0)}, Z'^{(j_0)}) > c_*/2 - 24\kappa$ .

By the coarse triangle inequality, for at least one point among  $X'^{(j_0)}, Z'^{(j_0)}$ , say  $X'^{(j_0)}$ , we have  $d_{R_1}^{\leq}(Y, X'^{(j_0)}) > c_*/4 - 13\kappa$ . Behrstock inequality gives  $d_Y^{\leq}(R_1, X'^{(j_0)}) \leq \kappa$ . □

**Lemma 2.5.** *Let  $\mathcal{W}$  be a composite windmill, and  $R_1, R_3$  be two osculators of  $\mathcal{W}$ . Assume  $\mathcal{W}_{j_0} \neq \emptyset$ , and let  $X_2 \in \mathcal{W}_{j_0}$ .*

*If  $R_3$  is of type neighbor and  $\mathcal{W}$  is  $(\frac{c_*}{2} - 20\kappa)$ -convex, then  $d_{R_1}(X_2, R_3) \leq c_*$ .*

*If  $R_3$  is of type gap, then  $d_{R_1}(X_2, R_3) \leq \Theta_P$ .*

*Proof.* If  $R_3$  is an osculator of neighbor type, then the result follows from Proposition 1.13.

If now  $R_3$  is an osculator of type gap, the proof is slightly more involved. There is  $i$ , and there are  $X, Z \in \mathcal{W}_i$  such that  $d_{R_3}^{\leq}(X, Z) > c_*/2 - 20\kappa$ .

Since  $\mathcal{W}_{j_0}$  is non-empty, and invariant for  $\Gamma_X$  and  $\Gamma_Z$ , we can apply Lemma 1.5 and find  $X^{(j_0)}, Z^{(j_0)} \in \mathcal{W}_{j_0}$  such that  $d_{R_3}(X^{(j_0)}, Z^{(j_0)}) \geq d_{R_3}^{\leq}(X, Z) - 4\kappa$  which is  $\geq c_*/2 - 24\kappa$ . By coarse triangular inequality, at least one of the quantities  $d_{R_3}(R_1, X^{(j_0)})$  and  $d_{R_3}(R_2, Z^{(j_0)})$  is greater than  $c_*/4 - 13\kappa$ . Say it is  $d_{R_3}(R_1, X^{(j_0)})$ . Behrstock inequality then gives that  $d_{R_1}(R_3, X^{(j_0)}) \leq \kappa$ , and again coarse triangular inequality gives  $d_{R_1}(X^{(j_0)}, X_2) \geq d_{R_1}(X_2, R_3) - \kappa$ . Since the first is bounded by the maximal convexity constant of  $\mathcal{W}$ , the result follows. □

### 2.3 The unfolding in the different coordinates

Given a composite windmill  $\mathcal{W}$ , we define its unfolding as follows.

An admissible set  $\mathcal{R}$  is one of the three following possibilities.

Consider  $\mathcal{R}_{gap}$  the set of osculators of type gap in  $\mathbb{Y}_{j_0}$ . If it is non-empty, we say  $\mathcal{R} = \mathcal{R}_{gap}$  is admissible, and is the only possible admissible set of osculators.

If  $\mathcal{R}_{gap}$  is empty, and if  $\mathcal{W}$  is  $(\frac{c_*}{2} - 20\kappa)$ -convex, then  $\mathcal{R} = \{G_W R\}$  for any choice  $R$  of osculator (necessarily of type neighbor) is admissible. There might be several admissible sets, and also there might be none.



If  $\mathcal{R}_{gap}$  is empty, and if  $\mathcal{W}$  is not  $\frac{c_*}{2}$ -convex, then the set  $\mathcal{R}$  of admissible osculators in this coordinate  $j_0$  is the empty set  $\mathcal{R} = \emptyset$ .

Here is an obvious lemma.

**Lemma 2.6.** *Let  $\mathcal{R}$  be a choice of an admissible set of osculators of  $\mathcal{W}$ . If  $\mathcal{R}$  is empty, then  $\mathcal{W}' = (\mathcal{W}_1, \dots, \mathcal{W}_m, G_W, j_0 + 1)$  is a composite windmill.*

We thus concentrate on the case where  $\mathcal{R}$  is non-empty.

Having made such a choice of set of admissible osculators  $\mathcal{R} \neq \emptyset$ , we then define, for all  $i$ ,  $\mathcal{W}'_i$  to be the union of all the images of  $\mathcal{W}_i$  by elements of the group  $G_{W'}$  generated by  $G_W \cup \{\bigcup_{R \in \mathcal{R}_a} \Gamma_R\}$ . The unfolding of  $\mathcal{W}$  is then

$$(\mathcal{W}'_1, \dots, \mathcal{W}'_m, G_{W'}, j_0 + 1).$$

In the case  $\mathcal{W}_{j_0}$  is empty, we include here a convexity result for an intermediate step in the construction: adding an admissible set of osculators  $\mathcal{R}$ , which produces a non-full composite windmill.

**Lemma 2.7.** *Assume that  $\mathcal{W}$  is a full composite windmill of principal coordinate  $j_0$ , with  $\mathcal{W}_{j_0} = \emptyset$ .*

*Let  $\mathcal{W}_{j_0}^s$  be a set  $\mathcal{R}$  of admissible osculators as defined above, assumed non-empty.*

*For all other coordinates, let  $\mathcal{W}_i^s = \mathcal{W}_i$ .*

*Then  $\mathcal{W}^s = (\mathcal{W}_1^s, \mathcal{W}_2^s, \dots, \mathcal{W}_m^s, G_W, j_0)$  is a non-full composite windmill of principal coordinate  $j_0$ . If moreover  $\mathcal{R}$  is the orbit of a neighbor osculator, and if  $\mathcal{W}$  is  $(\frac{c_*}{2} - 20\kappa)$ -convex, then  $\mathcal{W}^s$  is  $B$ -convex, for  $B = \frac{c_*}{2} + 10\kappa \leq \inf \mathcal{L}$ .*

*Proof.* If  $\mathcal{R} = \emptyset$ , there is nothing to prove. Consider the case of the orbit of a neighbor osculator. It suffices to check that  $\mathcal{W}_{j_0}^s (= G_W R)$  is convex in the sense that for all  $\gamma \in G_W$  and all  $i$  the set  $\mathbb{Y}_B^i(R, \gamma R)$  is in  $\mathcal{W}_i$ .

By the Greendlinger Property, given  $\gamma$ , there exists  $j$ , and  $Y_j \in \mathcal{W}_j$  such that  $d_{Y_j}(R, \gamma R) > \Theta_{Rot} - 2\Theta_P - \kappa$ , or  $R = \gamma R$  (if  $R$  is not active for all the shortening pairs of  $\gamma$ ).

Of course we consider only the first case of the alternative.

Assume that some  $Y \in \mathbb{Y}_i$  is in  $\mathbb{Y}_B^{(i)}(R, \gamma R)$ .

If  $Y \notin \text{Act}(Y_i)$ , then one can use a shortening pair at  $Y_i$  to reduce the length of  $\gamma$  in its principal tree, and this shortening pair gives  $\gamma'$  such that  $d_Y^{\triangleleft}(R, \gamma R) = d_Y^{\triangleleft}(R, \gamma' R)$ . Thus,  $Y \in \mathbb{Y}_B^{(i)}(R, \gamma' R)$  as well, and by performing this reduction sufficiently many times, we may assume that  $Y \notin \text{Act}(Y_i)$ .

By Lemma 1.4, either  $R$  or  $\gamma R$  approximates by  $\kappa$  the projection of  $Y_j$  on  $Y$ .

Say that  $d_Y^\triangleleft(\gamma R, Y_j) \leq \kappa$ . By osculation if  $Y \notin \mathcal{W}_i$ , one has  $d_Y^\triangleleft(Y_j, R) \leq \frac{c_*}{2}$ . Therefore, one has  $d_Y^\triangleleft(\gamma R, R) \leq d_Y^\triangleleft(\gamma R, Y_j) + d_Y^\triangleleft(R, Y_j) + \kappa \leq \frac{c_*}{2} + 2\kappa$  which is less than  $B$ .

If now  $d_Y^\triangleleft(R, Y_j) \leq \kappa$ , one has  $d_Y^\triangleleft(R, \gamma R)$  is within  $2\kappa$  from  $d_Y^\triangleleft(Y_j, \gamma R)$ , which equals  $d_{\gamma^{-1}Y}^\triangleleft(\gamma^{-1}Y_j, R)$ . Of course,  $Y \notin \mathcal{W}_i$  if and only if  $\gamma^{-1}Y \notin \mathcal{W}_i$ , hence, if it is the case, by osculation of  $R$ ,  $d_Y^\triangleleft(Y_j, \gamma R) \leq \frac{c_*}{2}$ , and  $d_Y^\triangleleft(\gamma R, R) \leq \frac{c_*}{2} + 2\kappa \leq B$ .

In the case where  $\mathcal{R}$  is the set of gap osculators, the proof is similar. Indeed, if  $R_1$  is a gap between  $X_1$  and  $Z_1$ , and  $R_2$  is a gap between  $X_2$  and  $Z_2$ , and if  $Y$  is between  $R_1$  and  $R_2$ , so that  $d_Y^\triangleleft(R_1, R_2) \geq c_* + 20(m-1)\kappa (= \mathcal{L}_{j_0}(j_0))$ , then  $Y$  is also between  $X_1$  (or  $Y_1$ ) and  $X_2$  (or  $Y_2$ ) so that, say,  $d_Y^\triangleleft(X_1, X_2) \geq c_* + 20(m-1)\kappa - 3\kappa$ . One can transfert  $X_2$  in the coordinate of  $X_1$  by Lemma 1.4, in  $\mathcal{W}$  (in the  $\Gamma_{X_2}$ -orbit of  $X_1$ ). The convexity of  $\mathcal{W}$  then shows that  $Y \in \mathcal{W}$ . □

The aim of the next sections is to prove the following.

**Proposition 2.8.** *If the set  $\mathcal{R}$  of admissible osculators is non-empty, then  $\mathcal{W}' = (\mathcal{W}'_1, \dots, \mathcal{W}_m, G_{\mathcal{W}'}, j_0 + 1)$  is a (full) composite windmill, and  $\mathcal{W}'_*$  can be chosen to contain  $\mathcal{W}_*$  (in other words,  $\mathcal{W}'$  is constructed over  $\mathcal{W}$ ).*

### 2.3.1 Unfolding a tree

**Proposition 2.9.** *(Principal coordinate tree)*

*Consider a full composite windmill  $\mathcal{W}$ , of principal coordinate  $j_0$ .*

*Let  $\mathcal{R} \neq \emptyset$  be an admissible set of osculators as defined in the previous section. If  $\mathcal{W}_{j_0} = \emptyset$ , let  $\mathcal{W}_{j_0}^s = \mathcal{R}$ , and otherwise let  $\mathcal{W}_{j_0}^s = \mathcal{W}_{j_0}$ .*

*There exists a  $G_{\mathcal{W}'}$ -tree  $T$ , bipartite, with black and white vertices, with an equivariant injective map  $\psi : T \rightarrow \mathcal{P}(\mathbb{Y}_{j_0})$  (the set of subsets of  $\mathbb{Y}_{j_0}$ ) that sends black vertices on images of osculators by  $G_{\mathcal{W}'}$ , and white vertices on images of  $\mathcal{W}_{j_0}^s$  by  $G_{\mathcal{W}'}$ , and that sent the neighbors (in  $T$ ) of the preimage of  $\mathcal{W}_{j_0}^s$  on  $\mathcal{R}$ .*

*Moreover, for any pair of distinct white vertices  $w_1, w_2$ , and any black vertex  $v$  in the interval between them (in  $T$ ), and any  $X_1 \in \psi(w_1), X_2 \in \psi(w_2)$ , one has  $d_{\psi(v)}(X_1, X_2) \geq \Theta_{\text{Rot}} - 2\Theta_P - \kappa$ .*

*Finally, if  $w_1, w_2$  are white vertices for which the path from a black vertex  $v$  starts by the same edge, then for any  $X_1 \in \psi(w_1), X_2 \in \psi(w_2)$ , one has  $d_{\psi(v)}(X_1, X_2) \leq 2\Theta_P + 3\kappa$ .*

*Proof.* Take a transversal  $\mathcal{R}^t$  of  $\mathcal{R}$  under the action of  $G_W$ . For each  $R \in \mathcal{R}^t$ , let  $(G_W)_R$  the subgroup of  $G_W$  generated by  $\bigcup_{X \in \mathcal{W} \setminus \text{Act}(R)} \Gamma_X$ .

Set  $T$  to be the Bass-Serre tree of the (abstract) graph of groups whose vertex groups are  $G_W$  and the groups  $\Gamma_R \times (G_W)_R, R \in \mathcal{R}^t$ , and the edges are the pairs  $(G_W, R), R \in \mathcal{R}^t$ , and the edge groups are the groups  $(G_W)_R$ .

Let  $\widetilde{G_{W'}}$  the fundamental group of this graph of groups. The group  $G_{W'}$  is a quotient of this group, since it is generated by  $G_W$  and the stabilizers of elements  $R$  of  $\mathcal{R}^t$ , which, by assumption (Definition 2.1), are direct sum of their rotation group with the groups  $(G_W)_R$ .

$T$  is a tree, endowed with a  $\widetilde{G_{W'}}$ -action, bipartite, and with an equivariant (with respect to  $\widetilde{G_{W'}} \twoheadrightarrow G_{W'}$ ) map  $\psi : T \rightarrow \mathcal{P}(\mathbb{Y}_j)$  that sends black vertices on images of elements of  $\mathcal{R}$  by  $G_{W'}$ , and white vertices on images of  $\mathcal{W}_{j_0}^s$  by  $G_{W'}$ .

We need to show that it is injective, and at the same time, we will show the estimate of the end of the statement.

Consider a path  $p$  of  $T$ , starting and ending at white vertex. Up to cyclic permutation, and up to the group action, we may assume that the path  $p$  starts at the vertex fixed by  $G_W$ , and its second vertex is fixed by some  $R_1 \in \mathcal{R}^t$ , and that its length is even.

Let us denote by  $p_0, p_1, \dots, p_N$  the consecutive vertices of  $p$ , and let  $X_{2i}$  be a choice of a element of  $\psi(p_{2i})$ , and  $R_{2i+1} = \psi(p_{2i+1})$ .

The monotonicity property in the coordinate  $j_0$  says that if  $d_Y(X, Z) \geq \Theta$  then  $d_W(X, Z) \geq d_W(X, Y)$ .

We will use it in an induction to establish that for all  $k$  odd, and all  $i$  in  $1 \leq i \leq \frac{N-k}{2}$  and all  $j$  in  $1 \leq j \leq \frac{k-1}{2}$ , one has

$$\begin{aligned} d_{R_k}(R_{k-2j}, R_{k+2i}) &\geq \Theta_{rot} - 2\Theta_P - \kappa \\ \forall X_s \in \psi(p_s), \quad d_{R_k}(X_{k-2j+1}, X_{k+2i-1}) &\geq \Theta_{rot} - 2\Theta_P - \kappa \end{aligned}$$

The case  $i, j = 1$  happens as follows. Choose  $k$ .

We first show how a black vertex separates two adjacent white vertices. Note that there is  $X'_{k+1} \in \psi(p_{k+1})$  that equals  $gX_{k-1}$  for some  $g \in \Gamma_{R_k} \setminus \{0\}$ . By convexity of  $\mathcal{W}_{j_0}^s$  (ensured by assumption, or by Lemma 2.7 in case  $\mathcal{W}_{j_0}$  is empty),  $d_{R_k}(X_{k+1}, X'_{k+1}) \leq \Theta_P$ . And by assumption on the rotating groups,  $d_{R_k}(X_{k-1}, X'_{k+1}) \geq \Theta_{Rot}$ . Thus,  $d_{R_k}(X_{k-1}, X_{k+1}) \geq \Theta_{Rot} - \Theta_P - \kappa$ , the second inequality.

By Lemma 2.5,  $d_{R_k}(X_{k+1}, R_{k+2}) \leq \Theta_P$  and  $d_{R_k}(X_{k-1}, R_{k-2}) \leq \Theta_P$ . By triangle inequality, we get  $d_{R_k}(R_{k-2}, R_{k+2}) \geq \Theta_{rot} - 2\Theta_P - \kappa$ . We have both inequalities.

Assume that the inequalities are proven for all  $(i, j)$  such that  $i + j \leq i_0$  (and for all  $k$ ), and let us choose  $k$  and  $(i, j)$  with  $i + j \leq i_0$ , and prove the inequality for  $(i + 1, j)$ .

Set  $Y = R_{k+2i}$ , and  $W = R_k$ . In the following we set either  $Z = R_{2i+k+2}$  or  $X_{2i+k+1}$ , and either  $X = R_{k-2j}$  or  $X = X_{k-2j+1}$ .

By the inductive assumption for  $k' = k + 2i$ ,  $i' = 1, j' = i$ , one has  $d_Y(W, Z) \geq \Theta_{rot} - 2\Theta_P - \kappa$ .

Also for  $k, i$  and  $j$  the induction gives  $d_W(Y, X) \geq \Theta_{rot} - 2\Theta_P - \kappa$ . Behrstock inequality then provides  $d_Y(W, X) \leq \kappa$  and therefore  $d_Y(X, Z) \geq \Theta_{rot} - 2\Theta_P - 3\kappa$ . This is still far above  $\Theta$ . One thus may apply the monotonicity property and obtain  $d_W(X, Z) \geq d_W(X, Y)$ . In other words,

$$d_{R_k}(R_{k-2j}, R_{k+2i+2}) \geq \Theta_{rot} - 2\Theta_P - \kappa.$$

The inequality is also proven for  $(i, j + 1)$  in the same manner, symmetrically. This finishes the induction.

In the end, we have obtained for  $i = N/2 - 1$ , and  $k = 1$ ,  $d_{R_1}(X_0, R_{N-1}) \geq \Theta_{Rot} - \Theta_P$ , and it follows that  $d_{R_1}(X_0, X_N) \geq \Theta_{Rot} - 2 \times \Theta_P - \kappa$ , which is the estimate of the statement.

If we assume that  $p$  is mapped to a loop,  $\mathcal{W}_{j_0}$  contains both  $X_0$  and  $X_N$ , and not  $R_1$  (it is an osculator), the convexity of  $\mathcal{W}_{j_0}$  imposes  $\Theta_{Rot} - 2 \times \Theta_P - \kappa \leq \Theta_P$ , meaning  $\Theta_{Rot} \leq \Theta_P + \kappa$ . and this contradicts our choice of  $\Theta_{Rot}$ .

It also follows from this analysis that if  $w_1, w_2$  are white vertices of  $T$  and  $v$  is a black vertex between them, then  $d_{\psi(v)}(X_1, X_2) \geq \Theta_{Rot} - 2\Theta_P - \kappa$  (in our induction above). A final use of Behrstock inequality provides that whenever the paths from  $v$  to a white vertex  $w_1$  has more than three edges, then if  $v'$  is the first black vertex after  $v$  on this path, and if  $X_1 \in \psi(w_1)$ , then  $d_{\psi(v)}(X_1, \psi(v')) \leq \kappa$ . It follows from that and Lemma 2.5 that if  $w_2$  is another white vertex  $w_1$  whose path from  $v$  starts at the same edge,  $d_{\psi(v)}(X_1, \psi(v')) \leq 2\Theta_P + 3\kappa$ .

□

The former proposition allows to define, for each element  $\gamma$  of  $G_{W'}$ , its principal coordinate, and its principal tree. Indeed, if  $\gamma \in G_{W'}$  not conjugated to  $G_W$ , the proposition shows that it is either loxodromic or the stabilizer of a black vertex on the tree  $T$ . Then we define its principal coordinate as  $j_0$  and its principal tree as  $T$ . If it is in  $G_W$ , or conjugate in it, its principal coordinate and its principal tree are defined inductively, according to the process of unfoldings of composite windmills.

### 2.3.2 Preservation of convexity

**Proposition 2.10.** *(Convexity of  $\mathcal{W}'$ )*

Let  $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_m, G_W, j_0)$  be a composite windmill (possibly non-full).

Assume that  $\mathcal{R}$  is an admissible set of osculators, and consider the sets  $\mathcal{W}'_i$  defined above. Let  $\mathcal{W}' = (\mathcal{W}'_1, \dots, \mathcal{W}'_m)$ .

If  $\mathcal{R}$  consists of the orbit of a neighbor, then  $\mathcal{W}'$  is  $c_*$ -convex.

If  $\mathcal{R}$  consists of gap osculators, then  $\mathcal{W}'$  is  $\mathcal{L}_{j_0+1}$ -convex.

The case of  $\mathcal{R} = \emptyset$  is trivial, so we assume it is not empty.

If  $\mathcal{R}$  consists of the orbit of a neighbor, let  $A_j = c_*$  for all  $j$ . If  $\mathcal{R}$  consists of gaps, let  $A_j = \mathcal{L}_{j_0}(j) + 20\kappa$  (which is less than  $\mathcal{L}_{j_0}(j+1)$ ).

Let  $X, Z \in \mathcal{W}'_i$ , consider  $Y \in \mathbb{Y}_{A(j)}^j(X, Z)$ .

Here is our main claim.

We will show that  $Y$  is a  $G_{W'}$ -translate of one of the following type of elements:

- $Y'$  for which there exists  $X_f, Z_f \in \mathcal{W}_{j_0}$  such that  $d_{Y'}^{\leq}(X_f, Z_f) \geq A(j) - 10\kappa$ ;
- $Y'$  for which there exists  $X_f \in \mathcal{W}_{j_0}$ , and  $R$  an osculator of  $\mathcal{W}$  in  $\mathcal{W}'_{j_0}$  such that  $d_{Y'}^{\leq}(X_f, R) \geq A(j) - 10\kappa$ ;
- $Y'$  for which there exists  $R_1, R_2$  osculators of  $\mathcal{W}$  in  $\mathcal{W}'_{j_0}$  such that  $d_{Y'}^{\leq}(R_1, R_2) \geq A(j) - 10\kappa$

We will then finish the proof with this claim established, but before that we will prove the claim.

*Transfert of  $X$  and  $Z$  to  $\mathbb{Y}_{j_0}$ .* In  $\mathcal{W}'$ , the groups  $\Gamma_X$  and  $\Gamma_Z$  preserve  $\mathcal{W}'_{j_0}$  which is not empty (it contains  $\mathcal{R}$ ). Therefore, by Lemma 1.5 there are  $X^{(j_0)}, Z^{(j_0)}$  in  $\mathcal{W}'_{j_0}$  such that  $d_Y^{\leq}(X^{(j_0)}, Z^{(j_0)}) \geq A(j) - 4\kappa$ .

*The interval in  $T$ .* Taking  $\psi^{-1}$  of  $X^{(j_0)}$  and of  $Z^{(j_0)}$  produce two vertices in the principal coordinate tree  $T$  of Proposition 2.9. More precisely, either one of  $X^{(j_0)}, Z^{(j_0)}$  is the image of a black vertex of  $T$ , or in the image of a white vertex of  $T$ . This thus give two vertices of  $T$  that we (slightly abusively) denote by  $\psi^{-1}(X^{(j_0)}), \psi^{-1}(Z^{(j_0)})$ .

If these vertices are adjacent, we have achieved the second point of the claim. If these vertices are the same, we have achieved the first point of the claim. If these vertices are different, both black with only one white vertex in the interval, we have achieved the third point of the claim.

Thus, we may assume that there is at least one black vertex of  $T$  in the open interval  $(\psi^{-1}(X^{(j_0)}), \psi^{-1}(Z^{(j_0)}))$ . Let  $R_1, \dots, R_N$  the images by  $\psi$  of these black vertices, in order starting from the side of  $\psi^{-1}(X^{(j_0)})$ .

By Proposition 2.9, we have for all  $i$ ,  $d_{R_i}(X^{(j_0)}, Z^{(j_0)}) > \Theta_{Rot} - 2\Theta_P - \kappa$ , which is  $> 50\kappa$ .

*Reduction to the case where  $R_i \in \text{Act}(Y)$*

If  $Y$  is equal to one of the  $R_i$  then we fall in the first possibility of the main claim. Thus, let us assume that  $Y$  is different from all the  $R_i$ .

We may assume that  $Y$  is in  $\text{Act}(R_i)$  for all  $i$ . Indeed if it was not, one could use an element of  $\Gamma_{R_i}$  to reduce the length of the path  $p$ , without changing the value of the projection distance  $d_Y^{\leq}(X^{(j_0)}, Z^{(j_0)})$  since  $\Gamma_{R_i}$  leaves  $d_Y^\pi$  invariant.

*Transfert of  $Y$  in  $\mathbb{Y}_{j_0}$ .* We may apply Lemma 1.4 again, and find an element  $Y^{(j_0)}$  in  $\mathbb{Y}_{j_0}$  (far in an orbit of  $\Gamma_Y$ ) such that, for all  $i$ , one has  $d_{R_i}^{\leq}(Y, Y^{(j_0)}) \leq 4\kappa$ .

*Position of  $Y^{(j_0)}$  in the order.* Fix  $0 < i \leq N$ . Since  $d_{R_i}(X^{(j_0)}, Z^{(j_0)}) > 50\kappa$ , either  $d_{R_i}(X^{(j_0)}, Y^{(j_0)})$  or  $d_{R_i}(Y^{(j_0)}, Z^{(j_0)})$  is larger than  $24\kappa$ .

All  $R_i$  are in  $\mathbb{Y}_{50\kappa}(X^{(j_0)}, Y^{(j_0)})$  therefore they satisfy the order property in this set, which coincide with the ordering of their indices. By this order property and Behrstock inequality, if for some  $i$  one has  $d_{R_i}(Y^{(j_0)}, X^{(j_0)}) > 5\kappa$ , then for all  $i' < i$ , one still has  $d_{R_{i'}}(Y^{(j_0)}, X^{(j_0)}) > 5\kappa$ . Similarly if  $d_{R_i}(Y^{(j_0)}, Z^{(j_0)}) > 5\kappa$  then for all greater  $i''$  the same holds.

Therefore we have three cases.

Either  $d_{R_1}(Y^{(j_0)}, X^{(j_0)}) \leq 5\kappa$  or  $d_{R_N}(Y^{(j_0)}, Z^{(j_0)}) \leq 5\kappa$ , or there exists  $i \geq 1$ , largest such that  $d_{R_i}(Y^{(j_0)}, X^{(j_0)}) > 5\kappa$  and  $i < N$ .

By symmetry, and translation by an element of  $G_{W'}$  the first and second case have same resolution. Let us treat the first one. By triangle inequality,  $d_{R_1}(Z^{(j_0)}, Y^{(j_0)}) > \Theta_{Rot} - 10\kappa - 2\Theta_P$  which is still greater than  $20\kappa$ .

Going back to  $Y$ :  $d_{R_1}^{\leq}(Z^{(j_0)}, Y) > 16\kappa$ . By Behrstock inequality,  $d_Y^{\leq}(Z^{(j_0)}, R_1) < \kappa$ , and finally by triangle inequality,  $d_Y^{\leq}(X^{(j_0)}, R_1) \geq A(j) - 2\kappa$ . We are in the second point of the claim if  $X^{(j_0)}$  is in a white vertex, and in the third point if it is a black vertex.

We thus turn to the case where there exists  $i \geq 1$ , largest such that  $d_{R_i}(Y^{(j_0)}, X^{(j_0)}) > 5\kappa$  and  $i < N$ .

One has

$$\begin{aligned} d_{R_{i+1}}(Y^{(j_0)}, Z^{(j_0)}) &> \Theta_{Rot} - 2\Theta_P - 10\kappa \\ d_{R_{i+1}}^{\leq}(Y, Z^{(j_0)}) &> \Theta_{Rot} - 2\Theta_P - 14\kappa \\ d_Y^{\leq}(R_{i+1}, Z^{(j_0)}) &\leq \kappa \end{aligned}$$

and

$$\begin{aligned} d_{R_i}(Y^{(j_0)}, X^{(j_0)}) &\geq 5\kappa \\ d_{R_i}^\triangleleft(Y, X^{(j_0)}) &\geq \kappa \\ d_Y^\triangleleft(R_i, X^{(j_0)}) &\leq \kappa \end{aligned}$$

So,  $d_Y^\triangleleft(R_i, R_{i+1}) \geq A(j) - 4\kappa$ . We have the third point of the claim, and the claim is established.

We need to finish the proof of the lemma. There are several cases to treat. The easiest is when the first case of the claim occurs.

In that case, if  $j = j_0$ ,  $Y'$  is actually a gap osculator, hence in  $\mathcal{W}'_{j_0}$ . If  $j \neq j_0$ , by convexity of  $\mathcal{W}$ , it is in  $\mathcal{W}_j$ .

Assume now that the second case occurs.

If  $R$  is of type neighbor, it simply contradicts Proposition 1.13.

If  $R$  is an osculator of type gap between  $X_0, X_1$ , and  $j = j_0$ , one easily gets that  $R$  is an osculator of type gap between  $X_f$  and either  $X_0$  or  $X_1$  (any one for which  $d_R(Y', X_\epsilon)$  is larger than  $\kappa$ , and by triangular inequality, there must be at least one). If  $j \neq j_0$ , we may use the same argument.  $Y' \in \text{Act}(R)$  therefore  $d_R^\triangleleft(Y', X_\epsilon)$  is larger than  $\kappa$  for either  $\epsilon = 0$  or  $1$ . Then,  $d_{Y'}^\triangleleft(R, X_\epsilon) < \kappa$  and by triangular inequality,  $d_{Y'}^\triangleleft(X_f, X_\epsilon) \geq A(j) - 12\kappa (> \mathcal{L}_{j_0}(j))$ . It follows by convexity of  $\mathcal{W}$  that  $Y' \in \mathcal{W}_j$ .

Finally, assume that the third case occurs.

Assume that  $R_2$  is an osculator of type gap, between  $X_0, X_1$ . Then, again with the same reasoning,  $Y' \in \text{Act}(R_2)$  and there is  $\epsilon$  for which it is in  $\text{Act}(X_\epsilon)$  and  $d_{Y'}^\triangleleft(R_2, X_\epsilon)$  is less than  $\kappa$ . Thus  $d_{Y'}^\triangleleft(R_1, X_\epsilon) \geq A(j) - 12\kappa$ , and we are back to the case 2 of the claim, with a slightly lower constant. The proof goes nevertheless through, and the desired conclusion holds.

Finally, assume that  $R_2$  is of type neighbor. Then both  $R_1, R_2$  are of type neighbor, and  $R_2 = \gamma R_1$  for some  $\gamma \in \Gamma_W$ . Let us rename  $R_1 = R$ , call  $i = i(Y')$ , and  $j$  the principal coordinate of  $\gamma$  (for the Greendlinger property). Let  $Z \in \mathcal{W}_j$  be the vertex of a shortening pair for  $\gamma$  for which  $Z \in \text{Act}(Y') \cap \text{Act}(R)$  (there exists one, otherwise one can reduce the length of  $\gamma$  in its principal tree by a shortening pair at  $Z$ ). Thus,  $d_Z^\triangleleft(R\gamma R) > \Theta_{Rot} - 2\Theta_P - 2\kappa$ .

Suppose  $d_{Y'}^\triangleleft(R, \gamma R) > c_* - 10\kappa$ . Then, there are two possible cases. Either  $d_{Y'}^\triangleleft(R, Z) > \frac{c_*}{2} - 6\kappa$  or  $d_{Y'}^\triangleleft(\gamma R, Z) > \frac{c_*}{2} - 6\kappa$  (or both).

In the first case,  $d_Z^\triangleleft(R, Y') \leq \kappa$ . Thus  $d_Z^\triangleleft(Y', \gamma R) > \kappa$ , and so  $d_{Y'}^\triangleleft(\gamma R, Z) < \kappa$ .

Recall that  $Z \in \text{Act}(R) \cap \text{Act}(Y')$ . Thus  $d_{Y'}^\triangleleft(Z, R) > c_* - 2\kappa$ , and  $Y' \in \mathbb{Y}_{c_* - 2\kappa}(Z, R)$ . Now let  $Z'$  any other element of  $\mathcal{W}$  in  $\text{Act}(R) \cap \text{Act}(Y')$ . By  $(\frac{c_*}{2} - 20\kappa)$ -convexity of  $\mathcal{W}$ , one has  $d_{Y'}^\triangleleft(Z, Z') \leq \frac{c_*}{2} - 20\kappa$  and therefore

$Y' \in \mathbb{Y}_{c_* - 2\kappa - \frac{c_*}{2} + 21\kappa}(Z', R)$ . In other words,  $Y' \in \mathbb{Y}_{\frac{c_*}{2} + 19\kappa}(\mathcal{W}, R)$  and this contradicts the fact that  $R$  is a neighbor.

In the second case, the situation is similar after composing by the automorphism  $\gamma^{-1}$ .

### 2.3.3 The unfolding is a windmill

**Proposition 2.11.** *If  $\mathcal{W} = (\mathcal{W}_1, \dots, \mathcal{W}_m, G_W, j_0)$  is a composite windmill, and if  $\mathcal{W}' = (\mathcal{W}'_1, \dots, \mathcal{W}'_m, G_{W'}, j_0 + 1)$  is defined as above (where  $j_0 + 1$  is taken modulo  $m$  in  $\{1, \dots, m\}$ ), then  $\mathcal{W}'$  is a composite windmill.*

*Moreover, the set  $\mathcal{W}'_*$  of the fifth point of the definition can be assumed to contain the set  $\mathcal{W}_*$  (in other words,  $\mathcal{W}'$  is constructed over  $\mathcal{W}$ ).*

The first three points follow by construction. The fourth point (convexity) is the result of Proposition 2.10. The sixth point is a consequence of Proposition 2.9. The same proposition introduces an action of  $G_{W'}$  on a tree  $T$  which is Bass-Serre dual to a presentation of  $G_{W'}$  as the fundamental group of a graph of group, with one vertex  $v_0$  carrying the group  $G_W$  and the other vertices  $v_{[R]}, [R] \in \mathcal{R}/G_W$ , adjacent to a single edge whose other end is  $v_0$ , carrying the group  $\Gamma_R \times (G_W)_R$ , if  $R$  is a representative of the orbit  $[R]$ .

## 2.4 Towers of windmills, and accessibility

### 2.4.1 Starting point

We start the process by selecting  $\mathcal{W}(0)$  to be a maximal collection of mutually inactive elements in  $\mathbb{Y}_*$ . Thus, whenever  $\mathcal{W}(0)_j \neq \emptyset$ , it is reduced to a single point.

We choose  $j_0 = 1$ . It is clear that  $\mathcal{W}(0)$  defines a composite windmill where for all  $i$ ,  $\mathcal{W}(0)_i$  is either empty or a singleton, and where  $G_W$  is the direct product of the groups  $G_X$ , for  $X \in \mathcal{W}(0)$  (there are at most  $m$  direct factors).

$\mathcal{W}(0)$  is  $\kappa$ -convex, and for all  $R$ , by maximality of  $\mathcal{W}(0)$ ,  $\text{Act}(R) \cap \mathcal{W}(0) \neq \emptyset$ . Recall that by choice,  $c_* > 25\kappa + 2\Theta$ , hence by Proposition 1.12, there exists a neighbor osculator in  $\mathbb{Y}_{\frac{c_*}{2} + 2m\kappa}(\mathcal{W}(0), R)$ . We set a counter  $N(0) = 0 \in \mathbb{N}$ , which will count the number of times we resort to a neighbor osculator.



### 2.4.2 The process

We need a process for choosing consistently among the possible neighbor osculators. Recall that we assumed  $\mathbb{Y}_*$  to be countable. Let us choose once and for all  $(X_n)_{n \in \mathbb{N}}$ , an infinitely redundant enumeration of elements of  $\mathbb{Y}_*$  (in the sense that every  $X \in \mathbb{Y}_*$  is equal to infinitely many  $X_n$ ).

We will define  $\mathcal{W}(k)$  for  $k$  a countable ordinal (not necessarily a number). We take the notation

$$\mathcal{W}(k) = (\mathcal{W}(k)_1, \dots, \mathcal{W}(k)_m, G_{\mathcal{W}(k)}, j_k).$$

The general process for defining  $\mathcal{W}(k+1)$  from  $\mathcal{W}(k)$  is the following.

If in the coordinate  $j_k$ , there is a gap osculator of  $\mathcal{W}(k)$  in  $\mathbb{Y}_{j_k}$  one defines the set of these gap osculators as  $\mathcal{R}(k)$  the set of admissible osculators at step  $k$ , and one unfolds  $\mathcal{W}(k)$  into  $\mathcal{W}'(k) = (\mathcal{W}'(k)_1, \dots, \mathcal{W}'(k)_m, G_{\mathcal{W}'(k)}, j_k + 1)$  (where  $(j_k + 1)$  is taken modulo  $m$ , in  $\{1, \dots, m\}$ ), which we call  $\mathcal{W}(k+1)$ . We set  $N(k+1) = N(k)$ .

If in the coordinate  $j_k$ , there is no gap osculator and if  $\mathcal{W}(k)$  is not  $(\frac{c_*}{2} - 20\kappa)$ -convex, then  $\mathcal{W}(k+1) = (\mathcal{W}(k)_1, \dots, \mathcal{W}(k)_m, G_{\mathcal{W}(k)}, j_k + 1)$ . We set  $N(k+1) = N(k)$ .

If in the coordinate  $j_k$ , there is no gap osculator and if  $\mathcal{W}(k)$  is  $(\frac{c_*}{2} - 20\kappa)$ -convex, then we consider any neighbor osculator  $R$  in  $\mathbb{Y}_{\frac{c_*}{2} + 2m\kappa}(\mathcal{W}(k), X_n) \cup \{X_n\}$  for  $n$  the minimal value above  $N(k)$  for which  $X_n \notin \mathcal{W}$ . We then set  $N(k+1) = N(k) + 1$ ,  $\mathcal{R} = G_{\mathcal{W}} R$ , then first

$$\mathcal{W}(k+1) = (\mathcal{W}(k)_1, \dots, \mathcal{W}(k)_m, G_{\mathcal{W}(k)}, i(R))$$

and then and one unfolds  $\mathcal{W}(k+1)$  into  $\mathcal{W}'(k+1)$ .

**Lemma 2.12.** *For each  $k$ , one and only one (except for the freedom in the choice of the neighbor) of these operations is possible.*

*Proof.* It follows from the maximality of  $\mathcal{W}(0)$ , that for all  $R$ ,  $\text{Act}(R) \cap \mathcal{W}(0) \neq \emptyset$ , hence also  $\text{Act}(R) \cap \mathcal{W}(k) \neq \emptyset$ . Thus, assuming  $\mathcal{W}(k)$  is  $(\frac{c_*}{2} - 20\kappa)$ -convex, one can find a neighbor osculator in  $\mathbb{Y}_{\frac{c_*}{2} + 2m\kappa}(\mathcal{W}(k), X_n) \cup \{X_n\}$  if  $X_n \notin \mathcal{W}(k)$ .  $\square$

**Lemma 2.13.** *In all these cases,  $\mathcal{W}(k+1)$  is still a composite windmill, constructed over  $\mathcal{W}(k)$ .*

*Proof.* This follows from Proposition 2.8 if the set of osculators is non-empty, and from Lemma 2.6 otherwise.  $\square$

Let us Zorn things up. If for all number  $s$ ,  $\mathcal{W}(k+s)$  is not  $(\frac{c_*}{2} - 20\kappa)$ -convex, then for countably many steps in the process, one never use a neighbor. Let us define  $\mathcal{W}(k+\omega)$  the direct union of the  $\mathcal{W}(k+s)$ ,  $s \in \mathbb{N}$ . It is a composite windmill, constructed over all  $\mathcal{W}(k+s)$  for  $s$  any integer. By Zorn Lemma, there is an ordinal  $\eta$  such that  $\mathcal{W}(k+\eta)$  is  $(\frac{c_*}{2} - 20\kappa)$ -convex.

### 2.4.3 Accessibility

**Lemma 2.14.** *For all  $R \in \mathbb{Y}_*$ ,  $R$  is eventually in  $\mathcal{W}(\eta)$ .*

*Proof.* Indeed, assume it is never in  $\mathbb{Y}_*$ . Each time  $\mathcal{W}(k)$  is  $(\frac{c_*}{2} - 20\kappa)$ -convex (and by the previous observation, this happens infinitely many times), we use a neighbor that is in  $\mathbb{Y}_{\frac{c_*}{2}+2m\kappa}(\mathcal{W}(k), X_n) \cup \{X_n\}$  for  $n$  the minimal value above  $N(k)$  for which  $X_n \notin \mathcal{W}$  (hence an integer). Because the sequence  $(X_n)_{\mathbb{N}}$  contains infinitely many times the element  $R$  (assumed to never be in  $\mathcal{W}(\eta)$ ), we will pick an element of  $\mathbb{Y}_{\frac{c_*}{2}+2m\kappa}(\mathcal{W}(k), R)$  infinitely many times. However,  $\mathbb{Y}_{\frac{c_*}{2}+2m\kappa}(\mathcal{W}(k), R) \subset \mathbb{Y}_{\frac{c_*}{2}+2m\kappa}(\mathcal{W}(0), R)$  and has at least one element less each times it is used by Proposition 1.11. Since by Proposition 1.10,  $\mathbb{Y}_{\frac{c_*}{2}+2m\kappa}(\mathcal{W}(0), R)$  is finite, this is a contradiction.  $\square$

## 2.5 End of the proof of the main Theorem 2.2

Applying the unfolding operation for each countable ordinal (while taking direct limits for each countable limit ordinal) allows to obtain a tower of windmills  $\mathcal{W}(k)$  (indexed by countable ordinals  $k$ ) with representatives  $\mathcal{W}(k)_*$  such that if  $k < k'$  then  $\mathcal{W}(k')$  is constructed over  $\mathcal{W}(k)$ . Applying Lemma 2.14 countably many times (for each element of  $\mathbb{Y}_*$ ), there exists  $k_{top}$  that is countable and that is the smallest ordinal such that  $\mathbb{Y} \subset \mathcal{W}(k)$ . Considering  $G_{\mathcal{W}(k_{top})}$ , we obtain the structure theorem desired, which is Theorem 2.2.

## 3 Conclusion, application to Dehn twists, and Theorem 1

Let  $\Sigma$  be an orientable closed surface of genus greater than 2. Consider  $\text{MCG}(\Sigma)$  its Mapping Class Group.

Bestvina Bromberg and Fujiwara produced a finite coloring of the set of simple closed curves of  $\Sigma$  such that two curves of same color intersect, and a finite-index normal subgroup  $G_0$  of  $\text{MCG}(\Sigma)$  that preserves the coloring.  $G_0$  is called the color preserving group. After refinement of the colors, we

actually may assume that the colors are in correspondance with the cosets of  $G_0$ . We denote the colors by  $\{1, \dots, m\}$ .

Let  $c$  and  $c'$  be a simple closed curves. If they intersect, the projection of  $c'$  on  $c$  is the family of elements in the arc complex of the annulus around  $c$  (that is the cover of  $\Sigma$  associated to  $c$ ) that come from lifts of  $c'$ . They are all disjoint. If  $c''$  is another simple closed curve intersecting  $c$ ,  $d_c^\pi(c', c'')$  is the diameter in the curve graph of the union of the projections of  $c'$  and  $c''$  on the annulus around  $c$ .

$d^\pi$  defines a composite projection system on the set of all (homotopy classes of) simple closed curves. Indeed, let  $\text{Act}(c)$  be the set of curves intersecting  $c$ . Clearly  $d_c^\pi$  is symmetric, and satisfies the separation. The symmetry in action, and the closeness in inaction are also direct consequences of definitions. The finite filling property is a consequence that all sequences of subsurfaces up to isotopy, increasing for inclusion, is eventually stationary.  $d_c^\pi$  satisfies the triangle inequality since it is a diameter of projections, and the Behrstock inequality [B], see also [Man] [Man2]. The properness is ensured by [BBF, Lemma 5.3]

We can now define two composite projection systems with composite rotating families. The first one is defined on  $\mathbb{Y}_*$  is the set  $\mathcal{S}$  of *all* homotopy classes of simple closed curves of  $\Sigma$ .

Let us define  $\mathbb{Y}_i$  to be the subset of this set of simple closed curve of color  $i$  in the Bestvina-Bromberg-Fujiwara coloring, and  $\mathbb{Y}_*$  their union. It is, as we just said, a composite projection system on which  $G_0$  acts by automorphisms.

Performing the construction of [BBF] and the choices as after Definition 1.2, we have constants  $\Theta, \kappa, c_*, \Theta_P, \Theta_{Rot}$ .

We select  $N_1$  such that all  $N_1$ -powers of Dehn twists in  $\text{MCG}(\Sigma)$  are in  $G_0$ . This is possible since there are only finitely many  $\text{MCG}(\Sigma)$ -orbits of simple closed curves in  $\Sigma$ , and  $G_0$  has finite index. Then we select  $N_2$  a multiple of  $N_1$  such that for all simple closed curve  $c$ , the Dehn twist  $\tau_c^{N_2}$  around  $c$  satisfies that  $d_c(c', \tau_c^{N_2} c') > \Theta_{Rot} + 2\Theta_P$  if  $c'$  is a curve of the same color than  $c$  (hence intersecting  $c$ ). Since  $d_c$  is comparable with  $d_c^\pi$ , by definition of the latter, there exists such an exponent  $N_2$ . Then it follows that, for all  $k \in \mathbb{N}$ , the collection  $\{\Gamma_c = \langle \tau_c^{kN_2} \rangle, c \in \mathcal{S}\}$ , is a composite rotating family for all  $k$ .

The second composite projection system is a sub-system, invariant for  $G_0$ , provided by the  $\text{MCG}(\Sigma)$ -orbit of a simple closed curve  $c_0 \in \mathcal{S}$ . Namely, the composite rotating family is the collection  $\{\Gamma_c, c \in \text{MCG}(\Sigma)c_0 \subset \mathcal{S}\}$ .

It is straightforward that both families are composite rotating families.

One can then apply Theorem 2.2. In the first case, one obtains that the group generated by the  $kN_2$ -th powers of all Dehn twists has a partially commutative presentation, which is the second point of Theorem 1. In the case of the second composite rotating family, one obtains that the group generated by all  $kN_2$ -th powers of all Dehn twists that are  $\mathrm{MCG}(\Sigma)$ -conjugated to  $\tau_{c_0}$  has a partially commutative presentation. This latter group is the normal closure of  $\tau_{c_0}^{kN_2}$  in  $\mathrm{MCG}(\Sigma)$ . We therefore obtained Theorem 1.

## Acknowledgements

The present work has been mostly developed during the visit of the author to the Mathematical Science Research Institute in Berkeley, during the thematic semester on Geometric Group Theory of the Fall 2016. The author is supported by the Institut Universitaire de France.

I wish to thank M. Bestvina, K. Bromberg, K. Fujiwara, J. Mangahas, J. Manning, and A. Sisto for discussions, and J. Tao, and S. Dowdall for organising an influential working seminar in MSRI. After I talked in MSRI on a first version of this work, which was performed on cone-offs of the blown-up projection complexes of [BBF], and was more in line with [DGO, §5], J. Mangahas suggested that I work directly in the language of projection complexes, which is indeed more natural for this situation, and allows a similar argument; this choice is in line with one of her work in progress with M. Clay and D. Margalit. I thank T. Brendle and D. Margalit for suggesting relevant references.

## References

- [B] J. Behrstock, Asymptotic geometry of the mapping class group and Teichmüller space. *Geom. Topol.* 10 (2006), 1523–1578.
- [BBF] M. Bestvina, K. Bromberg, K. Fujiwara, Constructing group actions on quasi-trees and applications to mapping class groups. *Publ. Math. Inst. Hautes Etudes Sci.* 122 (2015), 1–64.
- [BM] T. Brendle, D. Margalit, Personal communication, 2017.
- [CLM] M. Clay, C. Leininger, D. Margalit, Abstract commensurators of right angled Artin groups and Mapping Class Groups, *Math Research Letters*, 21 (3) 461–467 (2014).
- [Cou] R. Coulon, Partial periodic quotients of groups acting on a hyperbolic space, *Ann. Inst. Fourier, Grenoble*, 66, 5 (2016) 1774–1857.

- [DGO] F. Dahmani, V. Guirardel, D. Osin, Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces. *Mem. Amer. Math. Soc.* 245 (2017), no. 1156, v+152 pp.
- [Fa] B. Farb, Some problems on Mapping Class Groups and Moduli Spaces, in *Problems on Mapping Class Groups and Related Topics*, B. Farb Editor, *Proceedings of Symposia in Pure Mathematics*, vol.74, Amer. Math. Soc (2006) 11–56.
- [Fu] L. Funar, On the TQFT representations of the mapping class groups, *Pacific J. Math.* 188 (1999), no. 2, 251–274.
- [H] S. Humphries, Normal closures of powers of Dehn twists in Mapping Class Groups, *Glasgow, Math. J.* 34 (1992) 313–317.
- [I] N. Ivanov, Fifteen problems about Mapping Class Groups, in *Problems on Mapping Class Groups and Related Topics*, B. Farb Editor, *Proceedings of Symposia in Pure Mathematics*, vol.74, Amer. Math. Soc (2006) 71–80.
- [K] T. Koberda, Right-angled Artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups. *Geom. Funct. Anal.* 22 (2012), no. 6, 1541–1590.
- [Man] J. Mangahas, Uniform uniform exponential growth of subgroups of the mapping class group, *Geom. Funct. Anal.*, 19 (2010), 1468–1480.
- [Man2] J. Mangahas, A recipe for short-word pseudo-Anosovs, *Am. J. Math.*, 135 (2013), 1087–1116.
- [Mas] G. Masbaum, On powers of half-twists in  $M(0, 2n)$ , arXiv:1608.08449.
- [MM] H. A. Masur, Y. N. Minsky, Geometry of the complex of curves. I. Hyperbolicity, *Invent. Math.*, 138 (1999), 103–149.
- [S] C. Stylianakis, The normal closure of a power of a half Dehn twist has infinite index in the mapping class group of a punctured sphere, arXiv:1511.02912.

INSTITUT FOURIER, UNIV. GRENOBLE ALPES, CNRS, 38000 GRENOBLE, FRANCE  
e-mail: `francois.dahmani@univ-grenoble-alpes.fr`